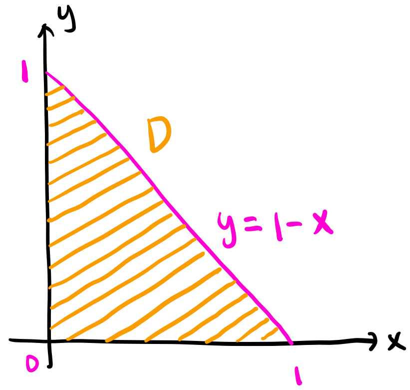
1. (10 points) The region D in the x-y plane is a triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Evaluate the integral

$$\int\int_D x^2y^2\,dA.$$

$D: 0 \le x \le 1,\ 0 \le y \le 1-x$



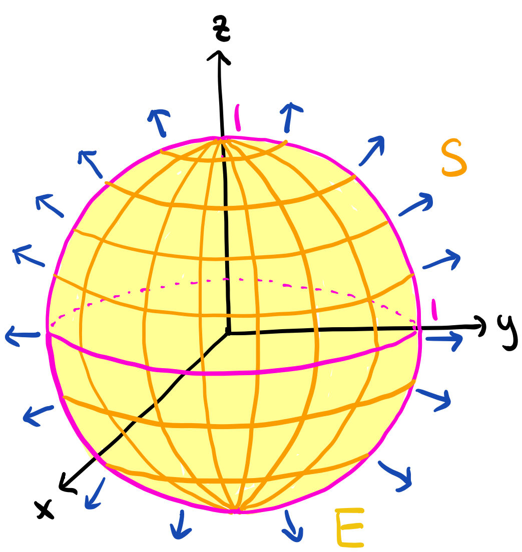
$$\iint_D x^2y^2\,dA = \int_0^1\int_0^{1-x} x^2y^2\,dy\,dx = \int_0^1 \frac{x^2y^3}{3}\Big|_{y=0}^{y=1-x} dx$$

$$= \frac{1}{3}\int_0^1 x^2(1-x)^3\,dx = \frac{1}{3}\int_0^1 x^2 - 3x^3 + 3x^4 - x^5\,dx$$

$$= \frac{1}{3}\left(\frac{x^3}{3} - \frac{3}{4}x^4 + \frac{3}{5}x^5 - \frac{x^6}{6}\right)\Big|_{x=0}^{x=1} = \boxed{\frac{1}{180}}$$

2. (10 points) Let $\mathbf{F}(x,y,z)$ be a vector field such that div$\mathbf{F} = 1+z$. Let S be the spherical surface $x^2+y^2+z^2=1$ with the normal $\mathbf{n}$ pointing outward. Evaluate the flux integral

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS.$$



*Symmetric about the xy, yz, zx planes

E : the solid ball $x^2+y^2+z^2 \le 1$

$\Rightarrow \partial E = S$ is positively oriented

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\text{div. thm}}{=} \iiint_E \underbrace{\mathrm{div}(\vec{F})}_{=\,1+z}\,dV$$

$$= \iiint_E 1+z\,dV = \iiint_E 1\,dV + \iiint_E \underbrace{z}_{\text{odd w.r.t. } z}\,dV$$

$$= \mathrm{Vol}(E) = \underbrace{\frac{4}{3}\pi\cdot 1^3}_{\text{volume of ball}} = \boxed{\frac{4\pi}{3}}$$

3. Let $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$ and let C be the curve $(x, y) = (e^t \sin t, e^{2t} \cos t)$, $0 \le t \le \pi$.

a) (4 points) Find a (potential) function f for the vector field $\mathbf{F}$ so that $\mathbf{F} = \nabla f$.

$$\vec{F}(x,y) = (x,y) \Rightarrow P = x,\ Q = y \Rightarrow \frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} \text{ is conservative.}$$

$$\vec{F} = \nabla f \Rightarrow P = f_x,\ Q = f_y.$$

$$\int P\,dx = \int x\,dx = \frac{x^2}{2}, \quad \int Q\,dy = \int y\,dy = \frac{y^2}{2}$$

$$\Rightarrow f(x,y) = \boxed{\frac{x^2}{2} + \frac{y^2}{2}}$$

(To get $f(x,y)$, collect all terms without duplicates)

b) (6 points) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

C is parametrized by $\vec{r}(t) = (e^t \sin t, e^{2t}\cos t)$ with $0 \le t \le \pi$.

$$\int_C \vec{F}\cdot d\vec{r} \underset{\text{Fund. thm}}{=} f(\vec{r}(\pi)) - f(\vec{r}(0)) = f(0, -e^{2\pi}) - f(0,1) = \boxed{\frac{1}{2}(e^{4\pi} - 1)}$$

4. Consider the line integral

$$\int_C \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

a) (4 points) Evaluate the line integral if C is the circle $x^2 + y^2 = 1$ in the counterclockwise sense.

This problem concerns the vortex field $\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

C is a circle centered at $(0,0)$ with CCW orientation.

$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \boxed{2\pi}$ (see Fact 3(2) in the Final exam facts note)

b) (6 points) Evaluate the line integral if C is the off-center ellipse $\frac{(x-2)^2}{16} + \frac{y^2}{25} = 1$ in the clockwise sense.

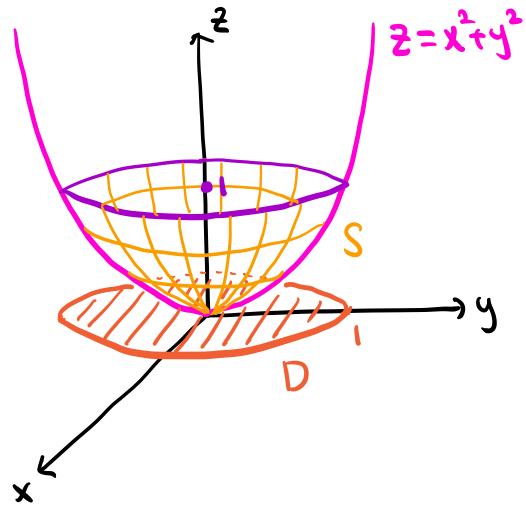
At $(0,0)$: $\frac{(x-2)^2}{16} + \frac{y^2}{25} = \frac{(0-2)^2}{16} + \frac{0^2}{25} = \frac{1}{4} < 1$.

C is a loop which encloses $(0,0)$ with CW orientation

$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \boxed{-2\pi}$ (see Fact 3(4) in the Final exam facts note)

5. Consider the parabolic surface (surface NOT volume) $z = x^2 + y^2$, $0 \le x^2 + y^2 \le 1$. Assume the density (mass per unit area) is constant and equal to 1.

a) (4 points) Find the total mass of the surface.



The surface S is parametrized by $\vec{r}(x,y) = (x, y, x^2+y^2)$.

The domain D is given by $x^2+y^2 \le 1$.

The density is $\rho(x,y,z) = 1$.

$$m = \iint_S \rho(x,y,z)\,dS = \iint_S 1\,dS = \iint_D |\vec{r}_x \times \vec{r}_y|\,dA.$$

$$\vec{r}_x = (1, 0, 2x),\ \vec{r}_y = (0, 1, 2y) \Rightarrow \vec{r}_x \times \vec{r}_y = (-2x, -2y, 1).$$

$$\Rightarrow m = \iint_D \sqrt{4x^2+4y^2+1}\,dA$$

In polar coordinates, D is given by $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

Jacobian

$$\Rightarrow m = \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_1^5 u^{1/2}\cdot\frac{1}{8}\,du\,d\theta$$

$$= \int_0^{2\pi} \left.\frac{u^{3/2}}{12}\right|_{u=1}^{u=5} d\theta = \frac{1}{12}\int_0^{2\pi} 5^{3/2}-1\,d\theta = \boxed{\frac{\pi}{6}(5^{3/2}-1)}$$

b) (6 points) Find the z-coordinate of the center of mass.

$$\bar{z} = \frac{1}{m}\iint_S z\,\rho(x,y,z)\,dS = \frac{1}{m}\iint_S z\,dS = \frac{1}{m}\iint_D (x^2+y^2)\,|\vec{r}_x \times \vec{r}_y|\,dA$$

$z = x^2+y^2$ on S

Jacobian

$$= \frac{1}{m}\iint_D (x^2+y^2)\sqrt{4x^2+4y^2+1}\,dA = \frac{1}{m}\int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1}\cdot r\,dr\,d\theta$$

$$= \frac{1}{m}\int_0^{2\pi}\int_1^5 \frac{u-1}{4}\cdot u^{1/2}\cdot\frac{1}{8}\,du\,d\theta = \frac{1}{m}\int_0^{2\pi}\int_1^5 \frac{u^{3/2}-u^{1/2}}{32}\,du\,d\theta$$

$u = 4r^2+1$

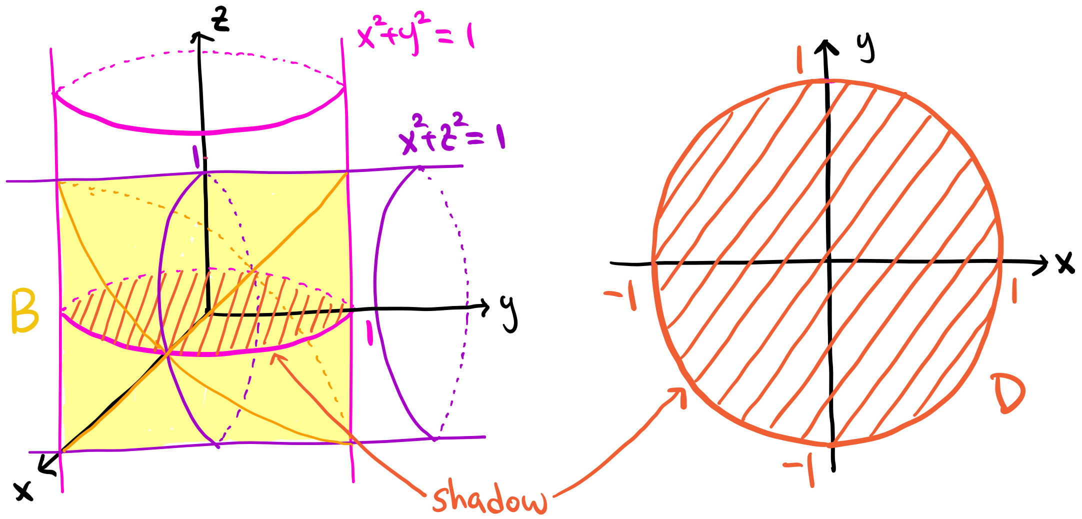
$$= \frac{1}{m}\int_0^{2\pi} \left.\frac{u^{5/2}}{80} - \frac{u^{3/2}}{48}\right|_{u=1}^{u=5} d\theta = \frac{6}{\pi(5^{3/2}-1)}\int_0^{2\pi} \frac{5^{5/2}+1}{120}\,d\theta = \boxed{\frac{5^{5/2}-1}{10(5^{3/2}-1)}}$$

6. Consider the solid cylinders $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$. Let B be the solid region of intersection of the two cylinders.

a) (6 points) Write down an integral of the form

$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} \int_{z=?}^{z=?} f(x, y, z)\, dz\, dy\, dx$$

for the volume of B. You must specify the bounds as well as the integrand $f(x, y, z)$.



$x^2+y^2=1 \Rightarrow y = \pm\sqrt{1-x^2}$, $x^2+z^2=1 \Rightarrow z = \pm\sqrt{1-x^2}$

The shadow D on the xy-plane : $-1 \leq x \leq 1$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

For each point on D : $-\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}$

$$\Rightarrow \text{Vol}(B) = \iiint_B 1\, dV = \boxed{\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\, dy\, dx}$$

b) (4 points) Find the volume of B, the region of intersection of the two cylinders.

$$\text{Vol}(B) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\, dy\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\, dy\, dx$$

$$= \int_{-1}^{1} 4(1-x^2)\, dx = 4x - \frac{4}{3}x^3 \Big|_{x=-1}^{x=1} = \boxed{\frac{16}{3}}$$

7. Consider the paraboloid surface $z = x^2 + y^2$.

a) (3 points) Find the equation of the plane tangent to the surface at $(1, 1, 2)$.

$z = x^2 + y^2 \Rightarrow x^2 + y^2 - z = 0 \rightsquigarrow$ a level surface of $f(x,y,z) = x^2 + y^2 - z$.

$\nabla f = (2x, 2y, -1) \Rightarrow \nabla f(1,1,2) = (2,2,-1)$ is a normal vector

The tangent plane at $(1,1,2)$ is given by

$\boxed{2(x-1) + 2(y-1) - (z-2) = 0}$

b) (7 points) Find the equation of the plane which is tangent to the paraboloid surface and contains the line $(x, y, z) = (t, 2 - 2t, -1)$.

Set $P = (a,b,c)$ to be the point of tangency.

A normal vector is $\nabla f(a,b,c) = (2a, 2b, -1)$

The line $\vec{l}(t) = (t, 2-2t, -1)$ has a direction vector $\vec{v} = (1,-2,0)$

and passes through $Q = \vec{l}(0) = (0, 2, -1)$

$\Rightarrow \nabla f(a,b,c)$ is perpendicular to $\vec{v}$ and $\overrightarrow{PQ}$

$\Rightarrow \nabla f(a,b,c) \cdot \vec{v} = 0$ and $\nabla f(a,b,c) \cdot \overrightarrow{PQ} = 0$

$$\Rightarrow \begin{cases} (2a, 2b, -1)\cdot(1,-2,0) = 0 \rightsquigarrow 2a - 4b = 0 \rightsquigarrow a = 2b \\ (2a,2b,-1)\cdot(-a, 2-b, -1-c) = 0 \rightsquigarrow -2a^2 + 4b - 2b^2 + 1 + c = 0 \quad (*) \end{cases}$$

$P = (a,b,c)$ is on the surface $z = x^2 + y^2 \Rightarrow c = a^2 + b^2 \underset{a=2b}{=} 5b^2$

$(*)$: $-2\cdot 4b^2 + 4b - 2b^2 + 1 + 5b^2 = 0 \Rightarrow -5b^2 + 4b + 1 = 0 \Rightarrow b = 1$ or $-\frac{1}{5}$.

Take $b = 1 \Rightarrow a = 2b = 2$, $c = a^2 + b^2 = 5 \Rightarrow P = (2,1,5)$.

A normal vector is $\nabla f(2,1,5) = (4,2,-1)$.

$\Rightarrow$ The plane is given by $\boxed{4(x-2) + 2(y-1) - (z-5) = 0}$

<u>Note</u> You can take $b = -\frac{1}{5}$ to get $P = (-\frac{2}{5}, -\frac{1}{5}, \frac{1}{5})$ and find

the plane equation $-\frac{4}{5}(x + \frac{2}{5}) - \frac{2}{5}(y + \frac{1}{5}) - (z - \frac{1}{5}) = 0$.

8. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$ be a vector field.

a) (5 points) Use trial and error to find a vector field $\mathbf{G}$ such that $\mathbf{F} = \text{curl}\mathbf{G}$.

Set $\vec{G} = (P, Q, R) \Rightarrow \text{curl}(\vec{G}) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$

$\vec{F} = (x, y, -2z) \Rightarrow x = R_y - Q_z, \ y = P_z - R_x, \ -2z = Q_x - P_y$.

Take $R = xy, \ Q = 0 \Rightarrow R_y - Q_z = x$.

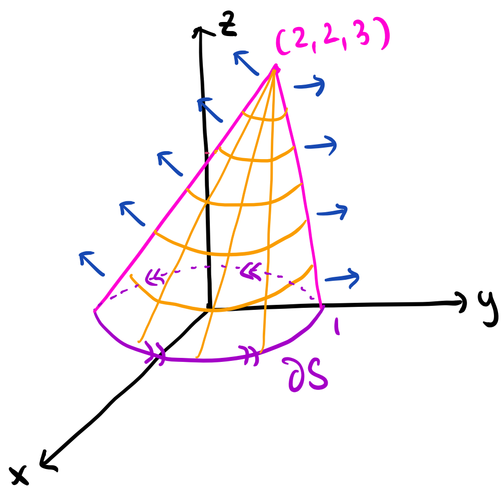
$y = P_z - R_x \Rightarrow y = P_z - y \Rightarrow P_z = 2y$

Take $P = 2yz \Rightarrow Q_x - P_y = 0 - 2z = -2z$

$\Rightarrow \vec{G} = \boxed{(2yz, 0, xy)}$

b) (5 points) Let S be the conical surface obtained by joining the circle $x^2 + y^2 = 1$ with $z = 0$ (circle in x-y plane) with the point $(2, 2, 3)$ by straight lines. The orientation of S is upward. Evaluate the flux integral

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, dS.$$



$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \text{curl}(\vec{G}) \cdot d\vec{S} \underset{\text{Stokes' thm}}{=} \int_{\partial S} \vec{G} \cdot d\vec{r}$

∂S is parametrized by

$\vec{r}(t) = (\cos t, \sin t, 0)$ with $0 \le t \le 2\pi$.

$\int_{\partial S} \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$.

$\vec{G}(\vec{r}(t)) = (0, 0, 2\cos t \sin t), \quad \vec{r}\,'(t) = (-\sin t, \cos t, 0)$

$\Rightarrow \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0 \Rightarrow \int_{\partial S} \vec{G} \cdot d\vec{r} = \boxed{0}$

Note You can also compute $\iint_S \vec{F} \cdot d\vec{S}$ using the divergence theorem by closing the bottom disk $x^2 + y^2 \le 1$ with $z = 0$.

9. (10 points) A particle moves along the helix $(x, y, z) = (\cos u, \sin u, u)$, which is parametrized by u, at constant speed equal to 1 in the upward direction. Its position at time $t = 0$ is $\mathbf{r}(0) = (1, 0, 0)$. Find the magnitude of the acceleration of the particle.

The speed is constant at 1

$\Rightarrow$ The time parameter t is equal to the arclength parameter

$$\Rightarrow t = \int_0^u |\vec{r}\,'(w)|\,dw$$

$$\vec{r}\,'(w) = (-\sin w, \cos w, 1) \Rightarrow |\vec{r}\,'(w)| = \sqrt{\sin^2 w + \cos^2 w + 1} = \sqrt{2}$$

$$\Rightarrow t = \int_0^u \sqrt{2}\,dw = \sqrt{2}u \Rightarrow u = \frac{t}{\sqrt{2}}$$

$$\Rightarrow \vec{r}(t) = \left(\cos\left(\frac{t}{\sqrt{2}}\right), \sin\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}}\right)$$

$$\Rightarrow \vec{r}\,'(t) = \left(-\frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cos\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \vec{r}\,''(t) = \left(-\frac{1}{2}\cos\left(\frac{t}{\sqrt{2}}\right), -\frac{1}{2}\sin\left(\frac{t}{\sqrt{2}}\right), 0\right)$$

The magnitude of the acceleration is

$$|\vec{r}\,''(t)| = \sqrt{\frac{1}{4}\cos^2\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{4}\sin^2\left(\frac{t}{\sqrt{2}}\right)} = \boxed{\frac{1}{2}}$$